Finite Math - Fall 2016

Lecture Notes - 9/19/2016

Section 3.3 - Future Value of an Annuity; Sinking Funds

Sinking Funds. We can turn the annuities picture around and ask how much we would need to deposit into an account each period in order to get the desired final value. It is simple to solve for PMT in the annuities formula to get

Definition 1 (Sinking Funds)**.**

$$PMT = FV \frac{i}{(1+i)^n - 1}$$

where all the variables have the same meaning as for annuities.

Example 1. *Let's revisit those new parents who are trying to save for their child's college and examine the more likely case that they will make payments into a savings account. They still want to save up* $\$80,000$ *and they have found an account that will pay* 8% *compounded quarterly. How much will they have to deposit every year in order to have a value of* $\$80,000$*?*

Solution. *In this case,* $i = \frac{r}{m} = \frac{0.08}{4} = 0.02$*,* $n = 4(17) = 68$ *and* $FV = \$80,000$*, so the required payment is*

$$PMT = \$80,000 \frac{0.02}{(1.02)^{68} - 1} = \$562.54.$$

Thus the parents would have to make a deposit of $\$562.54$ *every* 3 *months in order to have the desired* $\$80,000$ *after* 17 *years.*

Example 2. *A bond issue is approved for building a marina in a city. The city is required to make regular payments every* 3 *months into a sinking fund paying* 5.4% *compounded quarterly. At the end of* 10 *years, the bond obligation will be retired with a cost of* $\$5,000,000$*. How much will the city have to pay each quarter?*

Solution. $\$95,094.67$

Section 3.4 - Present Value of an Annuity; Amortization

Present Value of an Annuity. In the next concept, we will look at making a large deposit in order to have a fund which we can make constant withdraws from. We make an initial deposit, then make withdraws at the end of each interest period. We should have a balance of \$0 at the end of the predetermined amount of time the fund should last.

Example 3. *How much should you deposit into an account paying* 6% *compounded semiannually in order to be able to withdraw* $\$2000$ *every* 6 *months for* 2 *years? (At the end of the* 2 *years, there should be a balance of* $\$0$ *in the account.)*

Solution. *This problem is solved similarly to how the future value of an annuity was, except this time, instead of finding the future value of each deposit, we have to find the present value of each withdraw. We do this because we want to only deposit enough money to be able to withdraw the $2000 at the specified time. We can collect these again in a table:*

Withdraw	**Term Withdrawn**	**Number of times Compounded**	**Present Value**
$2000	1	1	$\$2000\left(1+\frac{0.06}{2}\right)^{-1} = \$2000(1.03)^{-1}$
$2000	2	2	$\$2000\left(1+\frac{0.06}{2}\right)^{-2} = \$2000(1.03)^{-2}$
$2000	3	3	$\$2000\left(1+\frac{0.06}{2}\right)^{-3} = \$2000(1.03)^{-3}$
$2000	4	4	$\$2000\left(1+\frac{0.06}{2}\right)^{-4} = \$2000(1.03)^{-4}$

So adding up the present values of all these will give us the amount of money we should deposit into the account now

$$D = \$2000(1.03)^{-1} + \$2000(1.03)^{-2} + \$2000(1.03)^{-3} + \$2000(1.03)^{-4} = \$7434.20$$

Of course, just as with finding the future value of an annuity, since these fit into a pattern, we can find a formula for it; and we actually do it in the exact same way as before by computing $1.03D - D$:

$$\begin{array}{rcllllllll} 1.03D & = & \$2000(1.03)^0 & + & \cancel{\$2000(1.03)^{-1}} & + & \cancel{\$2000(1.03)^{-2}} & + & \cancel{\$2000(1.03)^{-3}} & \\ -D & = & & - & \cancel{\$2000(1.03)^{-1}} & - & \cancel{\$2000(1.03)^{-2}} & - & \cancel{\$2000(1.03)^{-3}} & - \cancel{\$2000(1.03)^{-4}} \end{array}$$

This gives

$$1.03D - D = 0.03D = \$2,000 - \$2000(1.03)^{-4}$$

and solving for D

$$D = \$2,000\frac{1-(1.03)^{-4}}{0.03} = \$2,000\frac{1-\left(1+\frac{0.06}{2}\right)^{-4}}{0.06/2}.$$

This gives rise to the following formula

Definition 2 (Present Value of an Ordinary Annuity)**.**

$$PV = PMT\frac{1-(1+i)^{-n}}{i}$$

where

$$\begin{array}{rcl} PV & = & \textit{present value} \\ PMT & = & \textit{periodic payment} \\ i & = & \textit{rate per period} \\ n & = & \textit{number of payments (periods)} \end{array}$$

Note that the payments are made at the end of each period.

In the above formula, $i = \frac{r}{m}$, where r is the interest rate (as a decimal) and m is the number compounding periods per year and $n = mt$ where t is the length of time of the annuity. We can rewrite the formula with r and m instead of i

$$PV = PMT \frac{1 - \left(1 + \frac{r}{m}\right)^{-n}}{r/m}$$

Example 4. *How much should you deposit in an account paying* 8% *compounded quarterly in order to receive quarterly payments of* \$1,000 *for the next* 4 *years?*

Solution. \$13,577.71

An interesting application of this in conjunction with sinking funds is saving for retirement.

Example 5. *The full retirement age in the US is* 67 *for people born in* 1960 *or later. Suppose you start saving for retirement at* 27 *years old and you would like to save enough to withdraw* \$40,000 *per year for the next* 20 *years. If you find a retirement savings account (for example, a Roth IRA) which pays* 4% *interest compounded annually, how much will you have to deposit per year from age* 27 *until you retire in order to be able to make your desired withdraws?*

Solution. *First, we should figure out how much money we need to have in the account at the time we retire in order to be able to make the withdraws each year. For this situation, we have*

$$PMT = \$40,000, \quad m = 1, \quad r = 0.04, \quad t = 20, \quad n = mt = 20$$

and so the present value of the retirement account at the time of retirement needs to be

$$PV = \$40,000 \frac{1 - \left(1 + \frac{0.04}{1}\right)^{-20}}{0.04/1} = \$543,613.05.$$

So, now that we know how much we need to have in the account at the time of retirement, we can figure out how much we need to deposit into the savings account per year in order to achieve that amount in the 40 *years we have to save. To do this, we use the sinking fund formula. The future value here is the value we want at the time of retirement, so*

$$FV = \$543,613.05$$

and the other numbers are

$$r = 0.04, \quad m = 1, \quad t = 40, \quad n = mt = 40$$

Plugging this in the formula gives

$$PMT = \$543,613.05 \frac{0.04/1}{\left(1 + \frac{0.04}{1}\right)^{40} - 1} = \$5,720.71.$$

So we will have to deposit \$5,720.71 *per year from age* 27 *until retirement into this account in order to be able to withdraw* \$40,000 *per year for* 20 *years.*

Example 6. *Lincoln Benefit Life offered an ordinary annuity earning* 6.5% *compounded annually. If* $2,000 *is deposited annually for the first* 25 *years, how much can be withdrawn annually for the next* 20 *years?*

Solution. $10,688.87

Amortization. *Amortization* is the process of paying off a debt. The formula for present value of an annuity will allow us to model the process of paying off a loan or other debt. The reason the formula is the same is because receiving payments from your savings account is essentially the bank repaying you the money you loaned them by depositing it into a savings account.

Example 7. *Suppose you take out a* 5*-year,* $25,000 *loan from your bank to purchase a new car. If your bank gives you* 1.9% *interest compounded monthly on the loan and you make equal monthly payments, how much will your monthly payment be?*

Solution. *Since the loan was* $25,000 *and it is being paid off, the present value will be* $PV = \$25,000$*. The interest rate is* $r = 0.019$ *and is compounded monthly,* $m = 12$*. The loan lasts for* 5 *years, so we get*

$$\$25,000 = PMT\frac{1-\left(1+\frac{0.019}{12}\right)^{-60}}{\frac{0.019}{12}} = 57.19500PMT$$

Thus, solving for PMT *gives*

$$PMT = \frac{\$25,000}{57.19500} = \$437.10$$

which means our monthly payment would be $437.10 *for* 5 *years.*

We get the following formula

Definition 3 (Amortization)**.**

$$PMT = PV\frac{i}{1-(1+i)^{-n}}$$

where all the variables have the same meaning as for annuities.

Example 8. *If you sell your car to someone for* $2,400 *and agree to finance it at* 1% *per month on the unpaid balance, how much should you receive each month to amortize the loan in* 24 *months? How much interest will you receive?*

Solution. $PMT = \$112.98$, $I = \$311.52$

Amortization Schedules. Suppose you are amortizing a debt by making equal payments, but then decided to pay off the debt with one lump-sum payment. How do you find the "pay-off" balance of the debt? (E.g., you take out a 5-year loan with monthly payments for a car, but after 3-years of making payments you decide to just make one final payment to retire the debt.) This "pay-off" is very useful, even if you are not retiring the debt, but refinancing it. When refinancing a debt, you are essentially taking out a new loan to pay-off the previous debt, so you need to know how much unpaid balance remains on the account. When you are making payments into an amortization, at the beginning, a large part of your payment goes towards interest, while later, a larger part goes towards the unpaid balance.

We can see how much of each payment goes towards interest and how much towards unpaid balance by creating an *amortization schedule*.

Example 9. *Construct the amortization schedule for a* \$1,000 *debt that is to be amortized in six equal monthly payments at* 1.25% *interest per month on the unpaid balance.*

Solution. *The first step in this process is to compute the required monthly payment using the amortization formula*

$$PMT = \$1,000\frac{0.0125}{1-(1+0.0125)^{-6}} = \$174.03$$

Now, to figure out how much of the payment goes towards interest and how much towards unpaid balance, we compute the interest due at the end of the first month:

$$\$1,000(0.0125) = \$12.50$$

and so the amount of the payment that goes towards the unpaid balance is:

$$\$174.03 - \$12.50 = \$161.53.$$

Thus, the unpaid balance at the end of the first month is

$$\$1,000 - \$161.53 = \$838.47.$$

To compute the breakdown for the next month, we do the same thing, but with the new unpaid balance. The interest due at the end of month 2*:*

$$\$838.47(0.0125) = \$10.48$$

amount of payment towards unpaid balance:

$$\$174.03 - \$10.48 = \$163.55$$

and so the unpaid balance at the end of 2 *months is*

$$\$838.47 - \$163.55 = \$674.92.$$

Payment Number	**Payment**	**Interest**	**Unpaid Balance Reduction**	**Unpaid Balance**
0				\$1,000
1	\$174.03	\$12.50	\$161.53	\$838.47
2	\$174.03	\$10.48	\$163.55	\$674.92
3	\$174.03	\$8.44	\$165.59	\$509.33
4	\$174.03	\$6.37	\$167.66	\$341.67
5	\$174.03	\$4.27	\$169.76	\$171.91
6	\$174.03	\$2.15	\$171.91	\$0.00
Total	\$1,044.21	\$44.21	\$1,000	